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Stochastic Wess–Zumino–Witten model over a symplectic manifold

R. Léandre

Institut Elie Cartan, département de Mathématiques, Université de Nancy I, 54000 Vandoeuvre-les-Nancy, France

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Abstract

Over the path space of a symplectic manifold with end points in two Lagrangian submanifolds, we define a measure and a stochastic symplectic action in the simply connected case. We define a regularized Wess–Zumino–Witten Laplacian over the forms of finite degree over the path space. We perform a short time asymptotic near the critical points and find a limit Brownian harmonic oscillator: we can diagonalize it explicitly, and find the limit ground state of the Laplacian. We define a stochastic Witten complex, and its algebraic counterpart at the level of Chen forms.

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0. Introduction

Let us consider the loop space of a compact manifold: that is the space of smooth applications from the circle into the manifold. The propagation of a loop is related to the conformal field theory, because we consider a path integral over the set of applications from a Riemann surface with boundary into the manifold with the conformal group as a symmetry group. When there is no boundary, we consider random tori, and the integral over all the random tori for the given action gives the partition action of the theory. It is involved with a renormalization procedure and called the non-linear σ-model. In the flat case, it corresponds to the free field, and the measure lives over distributions. If we add some fermionic part to the non-linear σ-model, the partition function becomes the partition function of a supersymmetric non-linear σ-model and gives the index of some relevant operator over the loop space.

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The Wess–Zumino–Witten model is involved with the perturbation theory: we consider the exterior derivative over the loop space, its adjoint, and we perturb it by multiplying by a 1-form. When this 1-form is dF for a suitable functional, we call this functional the Wess–Zumino–Witten functional. We consider the associated Laplacian and its semi-group which is given by a path integral. In particular the trace (or the supertrace) of its semi-group is related to integral over random tori (see [JLW1,JLW2] for the case of a non-conformal field theory). After these previous works, Arai [Ar] constructed these random tori without renormalizing. The measure over the loop space lives over distributions. The integral over these random tori gives the index of some infinite-dimensional operator.

Witten [Wi] remarked that this perturbation of the free operator is related to the Morse theory, and this remark was fully exploited in [Bi7,HS] in finite dimension.

Our purpose is to try to give a rigorous formulation to the argument of Witten in infinite dimension.

For this purpose, we need to compute a Laplacian and therefore the adjoint of the exterior derivative. The applicant chosen in [JL2,L4,LR] is the Brownian bridge measure: this allows to compute infinite-dimensional operators rigorously over some loop spaces, and by an argument of deformation of the Brownian bridge measure, to compute their possible index by coming back to a flat model. In [ALR,DrR,LR], the cylinders are constructed with respect to the Ornstein–Uhlenbeck operator over some curved loop space: it is a beginning of the construction of the random torus. It did not require renormalization, unlike the conformal field theory.

The purpose of this paper is to try to generalize this construction to the case of the symplectic Morse theory over a path space. The model used here is quite different from the model used in [JL2,L4,LR], and is based on the work of Guilarte [Gui]. We give a stochastic interpretation of the Wess–Zumino–Witten model of Guilarte [Gui], using stochastic analysis, at least for the one-dimensional aspect of it, if we do not consider the propagation of the loop.

For this purpose, let us consider namely a compact symplectic manifold M and two compact Lagrangian submanifolds in transverse positions L and L'. Rabinowitz, Chaperon Conley, and Zehnder have introduced the space of paths going from L to L' in order to relate the topology of the full manifold to the structure of the intersection points of L and L'. This initial introduction was fully exploited by Floer later. How it can be seen?

Over the configuration space, we consider a closed 1-form σ which degenerates when we are over the constant loops. This form, at least when the path space is simply connected, can be integrated: $\sigma = dF$, for a suitable functional F, which is called the symplectic action. The global topology of the configuration space, which is involved with the topology of L, L' and M, is therefore related to the intersection points of L and L' by means of the Morse theory. This Morse theory is the purpose of the Floer homology. Morse theory, as it was pointed out by Witten, is related to the Wess–Zumino–Witten model by considering the complex $d + dF$, its dual $d^* + i_{dF}$ and the associated Laplacian Δ_F. Moreover, the Morse theory arises when we work over a small neighborhood of critical points: Guilarte pointed out that in a Morse system near the critical points over the path space, the Wess–Zumino–Witten Laplacian is a supersymmetric infinite-dimensional harmonic oscillator, whose structure is known. Near the constant paths, F is an iterated integral, and we meet the problem that the Hessian of F has an infinite number of negative eigenvalues as well as positive eigenvalues. A good understanding of this needs to introduce the Dirac sea [Gui].

The motivation of this paper is to explain some parts of Guilarte [Gui]. Namely, physicists use in order to compute the adjoint of an infinite-dimensional operator, the Lebesgue measure over the path space, which does not exist. We replace it by the Brownian bridge measure, which allows to define Sobolev spaces and other functional spaces. We introduce after [Bi2,JL1] a suitable tangent space chosen in order to get integration by parts formulas which depend on a parameter (see [L3,LN] when the parameter space is the manifold and not the Lagrangian submanifold).

This allows to define the bundle of forms over the path space: it is a fermionic Fock bundle. Moreover, the form σ which is a closed Chen form (see [JL1,L5]) can be integrated if we suppose that the path space is simply connected. We get a stochastic functional which checks for $\lambda > 0$ small enough:

$$E[\exp[\lambda|F|]] < \infty. \tag{0.1}$$

We introduce a connection which preserves the symmetry between L and L', the two Lagrangian submanifolds, which arise when we change the sense of the time. This allows to define a regularized exterior derivative $\mathrm{d_r}$ and a regularized Wess–Zumino–Witten operator $\mathrm{d_{rWZW}}$ after performing the scalar gauge transform associated to $\exp[\lambda F]$ over the stochastic regularized exterior derivative. The interest to choose a connection is that we can compute the adjoint of $\mathrm{d_{rWZW}}$ and the Wess–Zumino–Witten Laplacian Δ_{rWZW}.

In order to try to recover some topological information, we consider the Brownian bridge in small time. The probability law concentrates over the intersection points of the two Lagrangian submanifolds. We perform Bismut dilatations over the forms, which allow us to study the fluctuations over the intersection point of the model. We find Gaussian models, and the limit Wess–Zumino–Witten functional is strongly related to the area functional of Paul Lévy. These limit computations are very similar to those of [JL2,L4,LR] (for short time asymptotics, the reader can find surveys in [L1,K2,Wa]). The reader can find in [JLW1] computations which are similar in the domain of quantum field theory. But unlike these cases, the limit model is new and the limit probability measure is not related to finite-dimensional index theory [Bi3,Bi4]. In particular, we choose the couple $(x, y) \in T_L \times T_{L'}$ with the probability law $\exp(-\frac{1}{2}\|x - y\|^2)$. This seems to be new in the probabilistic literature.

At the limit, we find a Gaussian supersymmetric harmonic oscillator. Arai [Ar] has studied such operators in the domain of quantum field theory. We can study its behavior using a Morse system in infinite dimension in order to diagonalize it; it is strongly related to the study of the Ramer functional. The unique ground-state which is in L^2 is $\exp[\lambda F_l]$. In particular, there is no cohomology, except for the dimension 0. As it was pointed out by Guilarte [Gui], the good understanding of this limit harmonic oscillator needs the introduction of the Dirac sea, in order to get non-trivial cohomology groups at the limit; we will not speak of this problem, our goal being only to explain how we get this limit harmonic oscillator. This explains why the Floer homology is a middle homology theory of the path space [At].

In the third part, we speak of stochastics complexes: namely, the price to pay in order to be able to compute the adjoint of the exterior derivative is that we have only to consider an operator homotopic to the exterior derivative with its Wess–Zumino–Witten perturbation, using suitable connections (see [L2,L4]).

The problem to define a complex is strongly related to defining an anticipative Stratonovitch integral, because we have to take the covariant derivative of the parallel transport, which has a covariant derivative without finite variation.

Following the lines of Léandre [L5], we define a stochastic Witten complex over the configuration space, which is continuous for forms which belong to all the Sobolev spaces in the Nualart–Pardoux sense: they are involved with the regularity of the kernels of the derivatives. In particular, we meet the problem that $\exp[\lambda F]$ belongs only to some L^p, and not to all the Sobolev spaces. Therefore, it seems that the cohomology of the Witten complex is not immediately related to the cohomology of the configuration space, because the gauge transform is not in all the Sobolev spaces, as it can be checked on the limit model.

On the other hand, $\mathrm{d}F$ is a Chen form: over Chen forms, $\mathrm{d}F$ acts as a shuffle product [L6]. This allows us to define a Witten–Hochschild complex; we get a map between this algebraic model [MC] and the geometrical model using Chen iterated integrals. Over the introduced Hochschild space, we define Sobolev norms such that the Witten–Hochschild complex is continuous, and such that the map which to element of the Sobolev–Hochschild space associates the corresponding stochastic Chen iterated integral is continuous.

The reader can find in [IW] or in [El] an introduction to the stochastic differential geometry. Surveys about the Floer homology can be found in [At,Si,HZ].

1. Regularized Wess–Zumino–Witten model

Let M be a compact symplectic manifold, ω the symplectic form, and L and L' be compact Lagrangian submanifolds: ω is equal to zero over L and L'. We suppose that L and L' are transversal such that $L \cap L'$ is finite.

Let us set a Riemannian structure over M. $p_t(x, y)$ is the heat kernel associated to the Laplace–Beltrami operator. If we take the Riemannian structure $\langle . \rangle/\epsilon^2$, the heat kernel is transformed into $p_{t\epsilon^2}(x, y)$. Let $P_1(x, y)$ be the law of the Brownian bridge starting from x and arriving at y in time 1, and $P_{\epsilon^2}(x, y)$ be the law of the Brownian bridge associated to the Riemannian structure $\langle . \rangle/\epsilon^2$.

Let $P(L, L')$ be the space of continuous path starting from $x \in L$ and arriving at $y \in L'$ in time 1. We endow it with the probability measure

$$\frac{p_1(x, y)\,\mathrm{d}P_1(x, y)\,\mathrm{d}x \otimes \mathrm{d}y}{\int_{L\times L'} p_1(x, y)\,\mathrm{d}x\,\mathrm{d}y} = \mathrm{d}\mu_1(L, L'). \tag{1.1}$$

Let γ_t be a path and τ_t be the parallel transport from γ_0 to γ_t. A tangent vector is a path $\tau_t H_t$ such that:

- $H_0 \in T_{\gamma_0} L$,

- $\tau_1 H_1 \in T_{\gamma_1} L'$,
- H_t is a path with finite energy in $T_{\gamma_0} L$.

Let us remark that this tangent space is compatible with a time reversal of the path, which exchanges L and L'. Namely,

$$\tau_t H_t = \tau_t \tau_1^{-1} \tau_1 H_t \tag{1.2}$$

and $\tau_t \tau_1^{-1}$ is the parallel transport between γ_1 and γ_t, the path being reversed in time. So the calculus is compatible if we invert the role of L and L'.

Let us denote by T_γ this tangent space. We have the decomposition

$$T_\gamma = T_\gamma(L) \oplus T_{\gamma,\text{based}} \oplus T_\gamma(L'), \tag{1.3}$$

where

$$T_\gamma(L) = \{X_t = \tau_t(1-t)X_0;\ X_0 \in T_L\}, \tag{1.4}$$

$$T_\gamma(L') = \{X_t = \tau_t t \tau_1^{-1} X_1;\ X_1 \in L'\}, \tag{1.5}$$

$$T_{\gamma,\text{based}} = \{X_t;\ X_0 = X_1 = 0\}. \tag{1.6}$$

We decide that these three pieces of tangent spaces are orthogonal and we set as a Hilbert structure the following:
- over $T_\gamma(L)$:

$$\|X\|^2 = \|X_0\|^2, \tag{1.7}$$

- over $T_\gamma(L')$:

$$\|X\|^2 = \|X_1\|^2, \tag{1.8}$$

- over $T_{\gamma,\text{based}}$:

$$\|X\|^2 = \int_0^1 \|\,\mathrm{d}/\mathrm{d}s\, H_s\|^2 \,\mathrm{d}s. \tag{1.9}$$

We have an orthonormal basis of $T_{\gamma,\text{based}}$ given by Fourier expansion: if $n > 0$,

$$X_{n,i,t} = C\tau_t \frac{\sin(2\pi n t)}{n} \tau_{1/2}^{-1} e_i, \tag{1.10}$$

where e_i is an orthonormal basis of $T_{\gamma_{1/2}}$. If $n < 0$,

$$X_{n,i,t} = C\tau_t \frac{\cos(2\pi n t) - 1}{n} \tau_{1/2}^{-1} e_i. \tag{1.11}$$

We denote Y_i by

$$Y_{i,t} = \tau_t(1-t)e_i \tag{1.12}$$

and $Y'_{i,t}$ by

$$Y'_{i,t} = \tau_t t \tau_1^{-1} e_i. \tag{1.13}$$

In (1.12), e_i is an orthonormal basis of $T_L(\gamma_0)$ and in (1.13), e_i is an orthonormal basis of $T_{L'}(\gamma_1)$. In (1.10) and (1.11), we work over $T_{\gamma_{1/2}}$ in order that L and L' play a symmetric role.

Let $\sigma(\omega)$ be the Chen form:

$$\sigma(\omega)(X) = \int_0^1 \omega(\,\mathrm{d}\gamma_s, X_s). \tag{1.14}$$

Proposition 1.1. *$\sigma(\omega)$ is closed.*

Proof. The proof follows [L3, Theorem III.13] or [GJP].

Let us recall that, if σ is an r-form,

$$\begin{aligned}\langle \mathrm{d}\sigma, X^1, \ldots, X^{r+1}\rangle = &\sum_i (-1)^{i-1} \langle \mathrm{d}\langle \sigma, X^1, \ldots \hat{X}^i, \ldots, X^{r+1}\rangle, X^i\rangle \\ &+ \sum_{i<j} (-1)^{i+j} \langle \sigma, [X^i, X^j], X^1, \ldots \hat{X}^i, \ldots, \hat{X}^j, \ldots, X^{r+1}\rangle,\end{aligned} \tag{1.15}$$

where $\hat{}$ denotes the omission operator. When we take the exterior derivative of the Stratonovitch element $\omega(\,\mathrm{d}\gamma_s, \cdot)$, we have to take the derivative of $\omega(\gamma_s)$ and the derivative of $\mathrm{d}\gamma_s$. The derivative of $\mathrm{d}\gamma_s$ leads to the time covariant derivative of the vector field X_s. The derivative of $\omega(\gamma_s)$ leads to $\langle \nabla\omega(\gamma_s), X_s\rangle$. We add and substract the same term $\langle \nabla\omega(\gamma_s), \mathrm{d}\gamma_s\rangle$, and we recognize modulo sign the sum of the integral of the Stratonovitch differential of $\langle \omega(X_s, Y_s\rangle$ for the two vector fields X_s and Y_s and of the integral of $\mathrm{d}\omega(\,\mathrm{d}\gamma_s, X_s, Y_s)$. But $X_0, Y_0 \in T_l(\gamma_0)$ and $X_1, Y_1 \in T_{L'}(\gamma_1)$. Therefore

$$\int_0^1 \mathrm{d}\langle \omega(\gamma_s), X_s, Y_s\rangle = 0 \tag{1.16}$$

because L and L' are Lagrangian. Moreover, $\mathrm{d}\omega = 0$, because ω is a symplectic form. □

Theorem 1.2. *Let us suppose that $P(L, L')$ is simply connected. Then there exists a functional F such that $\mathrm{d}F = \sigma(\omega)$. Moreover, for $\lambda > 0$ small enough*

$$E[\exp[\lambda|F|]] < \infty. \tag{1.17}$$

Proof. Let $0 < t_1 < \cdots < t_n < 1$ be a finite dyadic subdivision of $[0,1]$, γ_s^n a polygonal path in $P(L, L')$: γ_s^n has a finite energy, and let $\gamma_s^{\mathrm{fixed}}$ be a polygonal path of reference. There exists a path of finite variation of diffeomorphism $\Psi_{t,s}(\cdot, \gamma^n)$ such that $\mathrm{d}/\mathrm{d}t\, \Psi_{t,s}(\cdot, \gamma^n)$ exists, and such that:

$$\Psi_{0,s}(\gamma_s^n, \gamma^n) = \gamma_s^n, \tag{1.18}$$

$$\Psi_{1,s}(\gamma_s^n, \gamma^n) = \gamma_s^{\text{fixed}}. \tag{1.19}$$

We will suppose in the sequel that $d(\gamma_{t_i}^n, \gamma_{t_{i+1}}^n) < \delta$ and that $\gamma_{t_i}^n$ belongs to a finite subset independent of n. Therefore the set of all possible such finite energy curves can be chosen in order that the following property is checked:

- The set of open balls of radius δ' $B(\gamma^n, \delta')$ for the uniform norm centered in some γ_s^n constitutes a recovering of $P(L, L')$.

There are at most C^n such γ_s^n. Let O_n be the event $\cup B(\gamma^n, \delta')$. The exponential inequality implies that if δ and δ' are small enough,

$$P(O_n^c \cap O_{n+1}) < P(\sup \mathrm{d}(\gamma_s, \gamma_t) > r) < \exp[-Cn], \tag{1.20}$$

the supremum is taken over the times s and t where $|s - t| < 1/n$ for a suitable r.

Let us choose such a curve γ^n: it has no bounded energy when $n \to \infty$. But if we rescale in order to get a curve $\tilde{\gamma}^n$ over the time interval [0,n], it has therefore a bounded derivative when $n \to \infty$. If we operate in time n, the family $\tilde{\Psi}_{t,s}(\cdot, \tilde{\gamma}^n), t \in [0, 1]$, has a bounded derivative in time s. Therefore, by coming back to time 1, the family $\Psi_{t,s}(\cdot, \gamma_n)$ has a derivative in time s bounded by Cn. Moreover, the diffeomorphism $\Psi_{1,s}(\cdot, \gamma^n)$ can be chosen in order to map a fixed small neighborhood of γ_s^n into a fixed small neighborhood of γ_s^{fixed} containing a small ball centered in these points with a given radius. Let us work on these small balls. $\Psi_{1,s}(\gamma_s, \gamma^n)$ belongs to these small balls centered in γ_s^{fixed}. Then we choose a geodesic deformation in order to arrive at γ_s^{fixed}, using exponential charts, and we operate in time [0, 1] using

$$\exp_{\gamma_s^{\text{fixed}}} \Psi_{1,s}(\gamma_s, \gamma^n). \tag{1.21}$$

Of course this deformation works if $d(\gamma_s, \gamma_s^n) < \delta'$ for a fixed δ' small enough. We get therefore a deformation $\Psi_{s,t}$ of a path belonging to the ball $B(\gamma^n, \delta')$ into γ_s^{fixed}. The process of deformation in time t is a semi-martingale, and the derivative in time t is a semimartingale which is bounded in time s in all the L^p. Since $\frac{\partial}{\partial x}\Psi_{t,s}$ is bounded, the Itô–Stratonovitch formula [Bi1] shows that the martingale part in time s of $\Psi_{t,s}$ is bounded in all the L^p and checks the exponential inequality. The finite energy part is bounded by Cn.

Let us suppose that $d(\gamma, \gamma^n) < \delta'$, and set

$$F(\gamma, \gamma^n) = \int_{[0,2]\times[0,1]} \sigma(\omega)\left(\mathrm{d}_s\Psi_{t,s}, \frac{\partial}{\partial t}\Psi_{t,s}\right). \tag{1.22}$$

From the previous considerations, we have, if $d(\gamma, \gamma^n) < \delta$,

$$E[\exp[\lambda|F(\gamma, \gamma^n)|]] < C\exp[C(\lambda)n], \tag{1.23}$$

where $C(\lambda) \to 0$ when $\lambda \to 0$.

The problem is that we can change the value of $F(\gamma, \gamma^n)$ if γ belongs to different small balls $B(\gamma^n, \delta)$ and $B(\gamma^{n'}, \delta)$ together. We use the support theorem of Ben-Arous and Léandre [BAL] or Aida et al. [AKS] in order to show in such a case, that almost surely,

$$F(\gamma, \gamma^n) = F(\gamma, \gamma^{n'}) \tag{1.24}$$

using the fact that $P(L, L')$ is supposed to be simply connected. We get, therefore, a functional $F(\gamma)$ over $P(L, L')$ by patching together all these functionals. Moreover,

$$E[\exp[\lambda |F|]] \leq \sum_n E[1_{O_n \cap O_{n-1}^c} \exp[\lambda |F|]] \leq \sum_n C^n \exp[C(\lambda) n] \exp[-Cn], \tag{1.25}$$

where the C^n arises from the number of ways to get curves γ_n.

This series does not converge, but if we operate in time ϵ^2 with the measure

$$\mathrm{d}\mu_{\epsilon^2}(L, L') = \frac{p_{\epsilon^2}(x, y)\,\mathrm{d}P_{\epsilon^2}(x, y)\,\mathrm{d}x \otimes \mathrm{d}y}{\int_{L \times L'} p_{\epsilon^2}(x, y)\,\mathrm{d}x\,\mathrm{d}y} \tag{1.26}$$

and if we take the functional F/ϵ^2, we get the upper bound

$$E_\epsilon \left[\exp\left[\lambda \frac{|F|}{\epsilon^2}\right]\right] \leq \sum_n C^n \exp\left[C(\lambda) \frac{n}{\epsilon^2}\right] \exp\left[-\frac{Cn}{\epsilon^2}\right], \tag{1.27}$$

where C does not depend on ϵ. Therefore,

$$E_\epsilon \left[\exp\left[\lambda \frac{|F|}{\epsilon^2}\right]\right] < \infty \tag{1.28}$$

for λ independent of ϵ and for ϵ small enough. By changing the Riemannian metric, we can come back to time 1, and we have (1.17) for ϵ small enough. In order to simplify the notation, we will operate in time 1.

In order to show that the H derivative [Gr] of the corresponding functional is $\sigma(\omega)$, we will consider a polygonal approximation of γ, γ^n, which is defined modulo a smooth cutoff. $F(\gamma_n)$ has a derivative in the classical sense, which is $\sigma(\omega)(\gamma_n)$, because in this case we work in finite dimension, and over smooth loop.

In order to define Sobolev spaces, we take a polygonal approximation of our path, and we denote by τ^n the parallel transport which is associated. If F^n is a functional which depends only on our polygonal approximation, we decide to take its derivative along the vector field $\tau_t^n H_t$. F belongs to all the Sobolev spaces with one derivative if it is the limit of F^n which depends only on the polygonal approximation in the Sobolev spaces with the tangent space given by $\tau_t^n H_t$. This definition has sense because the approximated tangent vector has divergences over the polygonal model by Léandre [L2] which tend to the divergences over the infinite-dimensional model. For higher-order Sobolev calculus, we proceed step-by-step and not globally as it is usual: a derivative of order r is derivable if there is a polygonal appoximation which is derivable in the previous sense. This needs to use some connections, because the r derivatives are r cotensors. We will not precise this choice of connection for the moment.

It remains to precise the smooth cutoff which allows to use the polygonal approximations of the path. Let g be a real function, which is constant equals to 1 near 0 and equals to 0 not very far from 0. The smooth cutoff is $G^n = \prod g(d(\gamma_{t_i}, \gamma_{t_{i+1}}))$ which tends to 1 in all the Sobolev spaces by the exponential inequality. □

Remark. Let us now precise how we can overcome the problem that $\sup d(\gamma_s, \gamma_s^n)$ is not smooth if γ_s^n has a bounded energy. We proceed as in [JL2] or in [L4]. Let h be a function from [0,1] into $[0, \infty]$ which is equal to $1/(-x+\delta)^p$ if $x \to \delta_-$ and $+\infty$ if $x \geq \delta$ and which is equal to 1 in a neighborhood of 0. Let us consider

$$H = \int_0^1 h(d(\gamma_s, \gamma_s^n))\, \mathrm{d}s \tag{1.29}$$

and

$$G(\gamma) = g(H) \tag{1.30}$$

from g from $[1, \infty]$ into [0,1] with a support into a small neighborhood of 0 and which is equal to 1 in 1. If $G \neq 0$, sup $\mathrm{d}(\gamma_s, \gamma_s^n) < \delta$". G moreover belongs to all the Sobolev spaces. Namely from the exponential inequality we deduce that:

$$P\left(\sup \frac{1}{(-d(\gamma_s, \gamma_s^n) + \delta)^+} \geq \frac{1}{\epsilon};\ H < C\right) \leq C(p)\epsilon^p \tag{1.31}$$

for all p. This shows us the property.

Let $\sigma_I(e)$ be a wedge product of $X_{n,i}(e)$, $Y_i(e)$ and $Y_i'(e)$ for a smooth system of sections e_i of $T_{\gamma_{1/2}}$, e_i of $T_{\gamma_0}(L)$ and e_i of $T_{\gamma_1}(L')$. We choose as core Λ the set of finite combination $\sigma_I(e)$ with cylindrical components. σ belongs to Λ if

$$\sigma = \sum_{I,e} F_I(e)\sigma_I(e). \tag{1.32}$$

Let us set

$$\begin{aligned} \mathrm{d_r wzw} = &\sum \nabla_{X_{n,i}(e)} \wedge X_{(n,i)}(e) \\ &+ \sum \nabla_{Y_i(e)} \wedge Y_i(e) + \sum \nabla_{Y_i'(e)} \wedge Y_i'(e) + \mathrm{d}F\wedge \end{aligned} \tag{1.33}$$

for any local orthonormal basis e_i of $T_{\gamma_{1/2}}(M)$, $T_{\gamma_0}(L)$ and $T_{\gamma_1}(L')$. We have to define in (1.33) a connection which preserves the metric. Over $T_\gamma(L)$, we take the pullback of the Levi-Civita connection of the Lagrangian manifold L over X_0 by the evaluation map $\gamma_. \to \gamma_0$. Over $T_\gamma(L')$, we take the pullback of the Levi-Civita connection of the Lagrangian manifold L' over X_1 by the evaluation map $\gamma_. \to \gamma_1$. Over $T_{\gamma,\mathrm{based}}$, we decide to write our vector $\tau_t\tau_{1/2}^{-1}H_t$ where H_t belongs to $T_{\gamma_{1/2}}$. We take the covariant derivative of H_t for the pullback of the Levi-Civita connection over M for the evaluation map $\gamma_. \to \gamma_{1/2}$. The connection preserves the Hilbert structure.

Since ∇ preserves the metric, we get formally

$$\begin{aligned} \mathrm{d_r^* wzw} = &\sum i_{X_{n,i}(e)}(-\nabla_{X_{n,i}(e)} + \operatorname{div} X_{n,i}(e)) \\ &+ \sum i_{Y_i(e)}(-\nabla_{Y_i}(e) + \operatorname{div} Y_i(e) \\ &+ \sum i_{Y_i'(e)}(-\nabla_{Y_i'(e)} + \operatorname{div} Y_i'(e)) + i_{\mathrm{d}F}, \end{aligned} \tag{1.34}$$

where the local orthonormal basis is chosen as before. We use that the adjoint of an exterior product by a vector X is an interior product i_X by the same vector.

Of course (1.34) has a sense only locally, because we need to use local sections of orthonormal basis e. In order to simplify the exposure, we will not write the partition of unity which appears in our computation, and which should appear in the divergences.

Let us recall the Bismut formula:

$$\nabla_X \tau_t = \tau_t \int_0^t \tau_u^{-1} R(\mathrm{d}\gamma_u, X_u)\tau_u. \tag{1.35}$$

This implies

$$\nabla_X \tau_t^{-1} = -\int_0^t \tau_u^{-1} R(\mathrm{d}\gamma_u, X_u)\tau_u \tau_t^{-1}. \tag{1.36}$$

We deduce from this that for $n > 0$:

$$\begin{aligned} \operatorname{div} X_{n,i} &= C \int_0^1 \langle \tau_s \cos(2\pi n s)\, \tau_{1/2}^{-1} e_i(\gamma_{1/2}), \delta\gamma_s\rangle \\ &\quad + C \int_0^1 \langle S_{X_{n,i}(s)}, \delta\gamma_s\rangle + \mathrm{O}(1/n), \end{aligned} \tag{1.37}$$

where S denotes the Ricci tensor. We also have too for $n < 0$:

$$\operatorname{div} X_{n,i} = C \int_0^1 \langle \tau_s \sin(2\pi n s)\, \tau_{1/2}^{-1} e_i(\gamma_{1/2}), \delta\gamma_s\rangle + \mathrm{O}(1/n). \tag{1.38}$$

We have

$$\operatorname{div} Y_i = \operatorname{div} e_i(\gamma_0) - \int_0^1 \langle \tau_s e_i(\gamma_0), \delta\gamma_s\rangle + 1/2 \int_0^1 \langle S_{Y_i(s)}, \delta\gamma_s\rangle. \tag{1.39}$$

We get an analoguous formula for $\operatorname{div} Y_i'$ by reversing time.

Let us now prove (1.39). We consider the family of Brownian motion over the manifold starting from $x \in L$ chosen at random $P_{\mathrm{open}}(L)$. By [LN,L2,L3], we have

$$\begin{aligned} &E_{P_{\mathrm{open}}(L)}[\langle \mathrm{d}F, Y_i\rangle f(\gamma_1)] + E_{P_{\mathrm{open}}(L)}[F\langle \mathrm{d}f(\gamma_1), Y_i\rangle] \\ &\quad = E_{P_{\mathrm{open}}(L)}[F \operatorname{div} Y_i\, f(\gamma_1)]. \end{aligned} \tag{1.40}$$

But $Y_{i,1} = 0$. Therefore $\langle \mathrm{d}f(\gamma_1), Y_i\rangle = 0$. We deduce that

$$E_{P_{\mathrm{open}}(L)}[\langle \mathrm{d}F, Y_i\rangle \mid \gamma_1 = y] = E_{P_{\mathrm{open}}(L)}[F \operatorname{div} Y_i \mid \gamma_1 = y] \tag{1.41}$$

and therefore (1.39) for $P(L, L')$.

Let us remark that if $F_{l,e}$ is a cylindrical functional, we have

$$|\langle \mathrm{d}F_{l,e}, X_{n,i}(e')\rangle| = \mathrm{O}(1/n) \tag{1.42}$$

and more precisely,

$$|\langle \mathrm{d}F_{l,e}, X_{n,i}(e')\rangle| \leq C/n, \tag{1.43}$$

where C is bounded in all the L^p.

Since in Λ, we consider finite combination of wedge products, we deduce that $\mathrm{d_{rWZW}}$ and $\mathrm{d^*_{rWZW}}$ are defined over Λ. The supercharge $Q = \mathrm{d_{rWZW}} + \mathrm{d^*_{rWZW}}$ is symmetric, therefore closable.

Definition 1.3. The regularized Wess–Zumino–Witten Laplacian is $\Delta_{\mathrm{rWZW}} = (\mathrm{d_{rWZW}} + \mathrm{d^*_{rWZW}})^2$.

D denotes the operation covariant derivative of a vector field over a loop.

Theorem 1.4. $\Delta_{\mathrm{rWZW}} = (\mathrm{d_{rWZW}} + \mathrm{d_{rWZW}})^2$ *is defined over* Λ, *and symmetric* ≥ 0. *It has therefore a self-adjoint extension.*

Proof. The fact that Δ_{rWZW} has a self-adjoint extension arises because Δ_{rWZW} is symmetric ≥ 0 densely defined.

Δ_{rWZW} can be split into different parts:

(a) $\mathrm{d}_r\,\mathrm{d}_r$: Let us recall that $\nabla_{X_n}\nabla_{X_m}X_{n',i}(e)$ has a behavior in $C/(|n|+1)(|m|+1)$, and that $\nabla_{X_n}\nabla_{Y_i}X_{n',i}(e)$ as well as the term obtained by inverting the order of the derivatives has a behavior in $C/(|n|+1)$. Therefore in $\mathrm{d_r}\,\mathrm{d_r}$, when we create two fermions, it has a component in $C/[(|n|+1)(|m|+1)]$ for the previous reasons and because

$$|\langle \mathrm{d}\langle \mathrm{d}F, X_n\rangle, X_m\rangle| \leq \frac{C}{(|n|+1)(|m|+1)} \tag{1.44}$$

because of (1.35). We do the convention that the derivative in Y_i or in Y_i' is enumerated by $i = 0$. Therefore if $\sigma \in \Lambda$, $\mathrm{d_r}\,\mathrm{d_r}\sigma$ is a series of forms which belongs to L^2.

(b) $\mathrm{d^*_r}\,\mathrm{d^*_r}$: If $\sigma \in \Lambda$, since σ is a finite sum, $\mathrm{d^*_r}\sigma$ is still a finite sum, and therefore $\mathrm{d^*_r}\,\mathrm{d^*_r}\sigma$ is still a finite sum. This term does not cause any difficulty.

(c) $\mathrm{d_r}\,\mathrm{d^*_r}$: If $\sigma \in \Lambda$, $\mathrm{d^*_r}\sigma$ is a finite sum, and therefore $\mathrm{d_r}\,\mathrm{d^*_r}$ is a series which belongs to L^2.

(d) $\mathrm{d^*_r}\,\mathrm{d_r}$: It causes some difficulties. Namely, there is an apparently diverging term, which is:

$$\sum_{n,i} -\nabla_{X_n(e_i)} i_{X_n(e_i)} \wedge X_n(e_i)\nabla_{X_n(e_i)}\sigma + \sum_{n,i} \operatorname{div} X_n(e_i) i_{X_n(e_i)} \wedge X_n(e_i)\nabla_{X_n(e_i)}\sigma. \tag{1.45}$$

But

$$\operatorname{div} X_n(e_i) = \int_0^1 \langle DX_n(e_i)_s, \delta\gamma_s\rangle + \mathrm{O}(1/n). \tag{1.46}$$

In $\nabla_{X_n(e_i)}\sigma$, we take the derivative either of a component of a cylindrical function which is a sum of expressions of the type $\alpha(\cos(2\pi n t_i) - 1)/n$ or $\beta \sin(2n\pi t_i)/n$ where α or β is a fixed random variable. We have the same property if we take the derivative of $\sigma_I(e)$. Therefore the only really diverging term in $\sum \operatorname{div} X_n(e_i) i_{X_n(e_i)} \wedge X_n(e_i)\nabla_{X_n(e_i)}\sigma$ is the term which arises from the stochastic integral in (1.46). But we recognize in the sum of this last term a stochastic integral of a deterministic process which is L^2 by a fixed random variable, or more precisely a sum of such expressions.

Let us study the second-order term in (1.45). $\nabla_{X_n(e_i)}\sigma$ is a polynomial expression in cylindrical term and in the parallel transport taken in a finite set of time mutiplied by $1/n$. The second derivative is by (1.35) a term in $1/n^2$. There is therefore no problem of convergence.

(e) $\mathrm{d_r}\,\mathrm{d}F\wedge$: In $\mathrm{d}F$, we create a series of $X_n(e)$ multiplied by terms of the order of $C/(|n|+1)$. In $\mathrm{d_r}$, either we derive σ, and create a series of $X_n \wedge X_m$ multiplied by term in $C/((|n|+1)(|m|+1))$ which converges in L^2, or we take the derivative of $\mathrm{d}F$, or more precisely of the component of $\mathrm{d}F$ which are $\int_0^1 \langle\omega(\mathrm{d}\gamma_s), X_n(e)_s\rangle$. Either we take the derivative of ω or of $X_n(e)_s$ in that expression along $X_m(e)$, which leads to a term in $C/(((|n|)+1)(|m|+1))$ which gives a series which converges in L^2. Or we take the derivative of $\mathrm{d}\gamma_s$. This leads to the series

$$\sum \int_0^1 \omega(DX_m(e_i), X_n(e_j))X_m(e_i) \wedge X_n(e_j). \tag{1.47}$$

But by an integration by part

$$\begin{aligned}\int_0^1 \omega(DX_m(e_i), X_n(e_j)) &= \mathrm{O}\left(\frac{1}{(|n|+1)(|m|+1)}\right) - \int_0^1 \omega(X_m(e_i), DX_m(e_j))\\ &= \mathrm{O}\left(\frac{1}{(|n|+1)(|m|+1)}\right) + \int_0^1 \omega(DX_n(e_j), X_m(e_i))\end{aligned} \tag{1.48}$$

by the antisymmetry of ω, and therefore the diverging term in (1.47) cancels because $X_m(e_i) \wedge X_n(e_j) = -X_n(e_j) \wedge X_m(e_i)$.

(f) $\mathrm{d}F \wedge \mathrm{d_r}$: We create a series of $X_n \wedge X_m$, whose all the components are in $C/((|n|+1)(|m|+1))$, which therefore converges.

(g) Let us study the tensorial terms: $\mathrm{d}F \wedge \mathrm{d}F$ is equal to zero as well as $i_{\mathrm{d}F} i_{\mathrm{d}F}$. It remains $\mathrm{d}F.i_{\mathrm{d}F} + i_{\mathrm{d}F}.\mathrm{d}F$. This term equals to $\|\mathrm{d}F\|^2$ which is in L^2.

(h) $\mathrm{d}F \wedge \mathrm{d_r^*}$: $\mathrm{d_r^*}\sigma$ is a finite sum. This term does not cause any difficulty as $i_{\mathrm{d}F}.\mathrm{d_r^*}$.

(i) $\mathrm{d_r^*} i_{\mathrm{d}F}$: $i_{\mathrm{d}F}\sigma$ is a finite sum, because σ is a finite sum. Therefore $\mathrm{d_r^*} i_{\mathrm{d}F}\sigma$ is a finite sum, which does not cause any difficulty.

(j) A difficult term which remains to be treated is $i_{\mathrm{d}F} \wedge \mathrm{d_r}$: but in $\mathrm{d_r}$, when we create an X_n, there is a term in C/n before it, and we annihilate a term in X_n in $i_{\mathrm{d}F}$, there is a term in C/n before it. Therefore the series converges. Moreover, $\mathrm{d_r} i_{\mathrm{d}F}$ does not cause any difficulty, because $i_{\mathrm{d}F}\sigma$ is a finite sum because σ is a finite sum. It remains to apply the integration by parts used in (1.47) in order to conclude.

(k) The last term that remains to be studied is $\mathrm{d}^*_{\mathrm{r}}\,\mathrm{d}F$. This leads to an apparently divergent term to treat. It is

$$-\sum\left\langle \mathrm{d}\int_0^1 \omega(\mathrm{d}\gamma_s, X_{n,i}), X_{n,i}\right\rangle + \sum\int_0^1 \omega(\mathrm{d}\gamma_s, X_{n,i})\,\mathrm{div}X_{n,i} = \mathrm{div}\tilde{X}_i, \qquad (1.49)$$

where $\tilde{X}_i$ is the vector over the based loop space given by

$$D\tilde{X}_{i,s} = \tau_s\left(-\int_0^s \omega(\mathrm{d}\gamma_u, \tau_u\tau_{1/2}^{-1}e_i) + C\right)\tau_{1/2}^{-1}e_i \qquad (1.50)$$

for some suitable random variable C constant in time s. The conclusion follows easily from the fact that the process $s \to \int_0^s \omega(\mathrm{d}\gamma_u, \tau_u)$ is adapted, because the Itô integral is equal in this case to the Skorohod integral. □

2. Limit Brownian harmonic oscillator

Instead of using the measure $\mathrm{d}\mu_1(L, L')$, we use the measure

$$\mathrm{d}\mu_{\epsilon^2}(L, L') = \frac{p_{\epsilon^2}(x, y)\,\mathrm{d}P_{\epsilon^2}(x, y)\,\mathrm{d}x \otimes \mathrm{d}y}{\int_{L\times L'} p_{\epsilon^2}(x, y)\,\mathrm{d}x\,\mathrm{d}y}. \qquad (2.1)$$

Over T_γ, we keep the splitting (1.3), but we change the metric and divide it by ϵ^2. $X_{n,i}$, Y_i, Y'_i are changed in $\epsilon X_{n,i}$, ϵY_i, $\epsilon Y'_i$. Let us introduce M_i the intersection points of L and L'.

In order to define $\mathrm{d}_{\epsilon\mathrm{rWZW}}$ and $\mathrm{d}^*_{\epsilon\mathrm{rWZW}}$, we perform the gauge transform associated with $\exp[\lambda(F/\epsilon^2)]$, by using (1.28). As forms, the vector fields $X_{n,i}$, Y_i, Y'_i are kept.

Let us introduce $\phi_i(\gamma_0, \gamma_1, \gamma_{1/2})$ a smooth cutoff function, which is equal to 1 if γ_0, γ_1, and $\gamma_{1/2}$ are close enough to M_i and such that $0 \le \sum\phi_i \le 1$. If ϕ_i is not equal to 0, and if the support of ϕ_i is small enough, there is a smooth section of orthonormal basis of $T_{\gamma_0}(L)$, $T_{\gamma_{1/2}}(M)$ and $T_{\gamma_1}(L')$.

Let us define Bismut's dilatation (cf [JL2,LR,L4]): let t_i be an enumeration of the rationals over [0,1]. Bismut's dilatation of a scalar functional is defined if γ_0 and γ_1 are close enough to a point M_i of L and L'. We denote in such cases by $\Pi\gamma_0$ and by $\Pi\gamma_1$ this intersection point (see [LR]).

We choose

$$F = f(\Pi\gamma_0)\prod_{I(n)}(f_i(\gamma_{t_i}) - f_i(\Pi\gamma_0)), \qquad (2.2)$$

$I(n)$ describes the set of part of cardinal n of Q. Let us suppose that the finite sum

$$\sum_I f_i(\Pi\gamma_0)\prod_I(f_{i,I}(\gamma_{t_i}) - f_{i,I}(\Pi\gamma_0)) \qquad (2.3)$$

is equal to zero. Since all the I are distinct, each component is equal to 0. This allows us to define Bismut's dilatation of such a functional if γ_0 and γ_1 are close enough to the intersection point M_i. If F satisfies (2.2),

$$B_\epsilon F = f(\Pi\gamma_0) \prod_{I(n)} \frac{f_i(\gamma_{t_i}) - f_i(\Pi\gamma_0)}{\epsilon}, \tag{2.4}$$

it can be extended by linearity.

Moreover:

$$\begin{aligned} f(\Pi\gamma_0) &\prod_{I(n)} (f_i(\gamma_{t_i}) - f_i(\Pi\gamma_0)) \\ &= f(\Pi\gamma_0) \prod_{I(n-1)} (f_i(\gamma_{t_i}) - f_i(\Pi\gamma_0)) f_n(\gamma_{t_n}) \\ &\quad - f(\Pi\gamma_0) f_n(\Pi\gamma_0) \prod_{I(n-1)} (f_i(\gamma_{t_i}) - f_i(\Pi\gamma_0)) \end{aligned} \tag{2.5}$$

By induction over n, we suppose that each cylindrical function $f(\Pi\gamma_0, \gamma_{t_1}, .., \gamma_{t_{n-1}})$ is the limit in L^2 of a sum of functions F of the shape (2.2) with a cardinal smaller than $n-1$. We deduce that $f(\Pi\gamma_0, \gamma_{t_1}, .., \gamma_{t_{n-1}}) f_n(\gamma_{t_n})$ is a limit of such a sum, and therefore that the set of functionals for which Bismut's dilatation is defined is dense in L^2 using the Stone–Weierstrass theorem, provided that γ_0 and γ_1 are close enough to an intersection point of the two manifolds.

If ϕ_i is not equal to 0, there are orthonormal bases of $T_\gamma\ X_{n,i}, Y_i, Y_i'$ which depend in a smooth way on $\gamma_{1/2}, \gamma_0, \gamma_1$. We deduce an orthonormal basis σ_I of the fermionic Fock space. Let $\sigma = \sum F_i\sigma_I$ when γ_0, γ_1, and $\gamma_{1/2}$ are close enough to M_i. We define Bismut's dilatation

$$B_\epsilon\sigma = \sum B_\epsilon(F_I)\sigma_I \tag{2.6}$$

if the F_I are finite combinations of functionals of the shape (2.2). Therefore B_ϵ is defined in L^2 of sections if γ_0, γ_1 and $\gamma_{1/2}$ are close enough to the intersection points of L and L'. If it is not the case, we perform no operations, and we stick together these two procedures by a smooth cutoff.

The idea is now to take $\epsilon \to 0$.

Let us define for that a limit model. We take a family of Gaussian spaces indexed by the finite set of points M_i of the intersection of L and L'. We choose the set of Brownian bridges starting from $x \in T_L(M_i)$ and going to $y \in T_{L'}(M_i)$: the Brownian bridge lives in $T_M(M_i)$. The law of (x, y) is the non-degenerate finite-dimensional law:

$$C \exp \tfrac{1}{2}[-\tfrac{1}{2}\|x-y\|^2]\,\mathrm{d}x \otimes \mathrm{d}y = \mathrm{d}Q_I(x, y). \tag{2.7}$$

This Gaussian law is non-degenerate because $T_L(M_i) \oplus T_{L'}(M_i) = T_M(M_i)$. Let $B_{t,\text{flat}}$ be the Brownian bridge starting from 0 and coming back to 0 in $T_M(M_i)$. The Brownian bridge between x and y satisfies:

$$\gamma_{t,\text{flat}} = x(1-t) + yt + B_{t,\text{flat}}. \tag{2.8}$$

The limit Wess–Zumino–Witten functional is

$$F_l = \frac{1}{2}\int_0^1 \omega(\mathrm{d}\gamma_{s,\mathrm{flat}}, \gamma_{s,\mathrm{flat}}). \tag{2.9}$$

Let us compute $\mathrm{d}F_l$. The tangent space of a flat path is given by a curve $X_t = X(1-t) + Yt + \int_0^1 H_s' \,\mathrm{d}s$, where $\int_0^1 H_s' \,\mathrm{d}s = 0$. We have

$$\langle \mathrm{d}F_l, X\rangle = \frac{1}{2}\int_0^1 \omega(\mathrm{d}\gamma_{s,\mathrm{flat}}, X_s) + \frac{1}{2}\int_0^1 \omega(X_s', \gamma_{s,\mathrm{flat}})\,\mathrm{d}s. \tag{2.10}$$

We integrate by parts and we use the fact that ω is antisymmetric, and that L and L' are Lagrangian. We deduce that

$$\langle \mathrm{d}F_l, X\rangle = \int_0^1 \omega(\mathrm{d}\gamma_{s,\mathrm{flat}}, X_s). \tag{2.11}$$

Since ω is antisymmetric, we can consider in (2.9) a double Itô integral or a double Stratonovitch integral. We can write ω in M_i in a suitable orthonormal basis as a finite set of matrices

$$\begin{pmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{pmatrix}.$$

The limit Wess–Zumino–Witten functional can be split into three parts:

(a) A sum of Levy areas $\lambda_i/2(\int_0^1 \mathrm{d}B_s^1\, B_s^2 - \int_0^1 B_s^2\,\mathrm{d}B_s^1)$. In order to understand this contribution, let us write

$$\mathrm{d}B_s^1 = C\sum \lambda_n \cos(2\pi ns)\,\mathrm{d}s + C\sum \mu_n^1 \sin(2\pi ns)\,\mathrm{d}s, \tag{2.12}$$

$$B_s^2 = C\sum -\mu_n^2 \frac{\cos(2\pi ns) - 1}{n} + C\sum \lambda_n^2 \frac{\sin(2\pi ns)}{n}. \tag{2.13}$$

We have

$$\int_0^1 \mathrm{d}B_s^1 B_s^2 = C\sum\left(-\frac{\lambda_n^1\mu_n^2}{n} + \frac{\lambda_n^2\mu_n^1}{n}\right). \tag{2.14}$$

After using an integration by parts, we deduce a system of Morse coordinates for this part of the limit symplectic action

$$\int_0^1 \mathrm{d}B_s^1 B_s^2 = C\left\{\sum \frac{(\mu_n^1 + \lambda_n^2)^2}{n} - \sum \frac{(\mu_n^1 - \lambda_n^2)^2}{n} + \sum \frac{(\mu_n^2 - \lambda^1)^2}{n} - \sum \frac{(\mu_n^2 + \lambda_n^1)^2}{n}\right\}. \tag{2.15}$$

Let us set

$$Y_n^1 = \int_0^1 \sin(2\pi ns)\,\mathrm{d}B_s^1 + \int_0^1 \cos(2\pi ns)\,\mathrm{d}B_s^2, \tag{2.16}$$

$$Y_n^2 = \int_0^1 \sin(2\pi ns)\,\mathrm{d}B_s^2 + \int_0^1 \cos(2\pi ns)\,\mathrm{d}B_s^1, \tag{2.17}$$

$$Z_n^1 = \int_0^1 \sin(2\pi ns)\,\mathrm{d}B_s^1 - \int_0^1 \cos(2\pi ns)\,\mathrm{d}B_s^2, \tag{2.18}$$

$$Z_n^2 = \int_0^1 \sin(2\pi ns)\,\mathrm{d}B_s^2 - \int_0^1 \cos(2\pi ns)\,\mathrm{d}B_s^1. \tag{2.19}$$

The system of Y_n^1, Y_n^2, Z_n^1, Z_n^2 consist of a system of independent Gaussian variables of the same expectation and with the same variance. Moreover

$$\int_0^1 \mathrm{d}B_s^1 B_s^2 = C\left\{\sum \frac{(Y_n^1)^2 - C}{n} - \sum \frac{(Z_n^1)^2 - C}{n} \right.$$
$$\left. + \sum \frac{(Z_n^2)^2 - C}{n} - \sum \frac{(Y_n^2)^2 - C}{n}\right\}. \tag{2.20}$$

(b) Let us study the interacting term between the infinite-dimensional part and the finite-dimensional part. An integration by parts shows

$$\int_0^1 \omega(y - x, B_{t,\mathrm{flat}}) + \int_0^1 \omega(\mathrm{d}B_{t,\mathrm{flat}}, x(1-t) + yt)$$
$$= \int_0^1 \omega(y - x, B_{t,\mathrm{flat}}) - \int_0^1 \omega(B_{t,\mathrm{flat}}, x - y) = 2\int_0^1 \omega(y - x, B_{t,\mathrm{flat}}) \tag{2.21}$$

because L and L' are Lagrangians. Therefore in the system of Y_n^1, Y_n^2, Z_n^1, and Z_n^2 after writing the symplectic form in the simplest way, the interacting terms are in C/n.

(c) The finite-dimensional part is finite and does not cause any problem.

Let H be a Hilbert space and an abstract Wiener space associated to this Hilbert space, and O be a symmetric Hilbert–Schmidt operator $a_{i,j}$ after choosing an orthonormal basis of H. Then

$$O(w) = \sum (a_{i,j} Z_i Z_j - \delta_{i,j} a_{i,j}) \tag{2.22}$$

is called the Ramer functional associated to the Hilbert–Schmidt operator O [Ra2, AB]: the Z_i denotes the system of independent centered Gaussian variables associated to the orthonormal basis. $O(w)$ is in L^2 and is independent of the choice of the Hilbert basis.

From the previous considerations, we deduce that F_l is the sum of a constant and of a functional of the type (2.22). In particular, if $\lambda > 0$ small enough, then

$$E[\exp[\lambda|F_l|]] < \infty. \tag{2.23}$$

Namely, we can split F_l in a sum of Lévy areas which check (2.23), finite-dimensional quadratic terms which check (2.23) and an interacting term $\int_0^1 \omega(x-y, B_{t,\text{flat}})$ which is smaller than $C\|x-y\| \sup|B_{t,\text{flat}}|$. But if λ is small enough, we can use the exponential inequality in order to show that $E_B[\exp[\lambda\|x-y\| \sup|B_{t,\text{flat}}|] \le \exp[C(\lambda)\|x-y\|^2]$, where $C(\lambda) \to 0$ when $\lambda \to 0$. This allows us to conclude.

In order to simplify the exposure, we will suppose that $\lambda = 1$.

At the limit, we consider the operator d_l, the Shigekawa complex of the limit Gaussian model [Ar]. We add the Wess–Zumino–Witten term $\mathrm{d}F_l$ (see [Ar]). We get an operator $\mathrm{d}_{\mathrm{lWZW}} = \mathrm{d}_l + \mathrm{d}F_l\wedge$. Its adjoint can be computed: it is $\mathrm{d}_l^* + i_{\mathrm{d}F_l}$. The only difference in the computation of d_l^* in [Sh] is the finite-dimensional Gaussian term $\exp[-\frac{1}{2}\|x-y\|^2]$. The divergence of the constant vector $X \in T_L(M_i)$ is $\langle X, x-y\rangle$ and of $Y \in T_{L'}(M_i)$ is $\langle Y, y-x\rangle$.

We see that $d\mu_{\epsilon^2}(L, L')$ tends in law to $\sum \alpha_i \delta_{M_i}$ for some positive reals α_i. $\sum \alpha_i = 1$, $\alpha_i > 0$. As limit model, we choose the point M_i with the law α_i and around these points the previous Gaussian law in order to analyze the fluctuations.

Let us now precise what we mean by a theorem in law. The fiber is isomorphic to $\Lambda T_L(\gamma_0) \wedge \Lambda_{L'}(\gamma_1) \wedge \Lambda_{\gamma_{1/2}}(H)$, where the last expression denotes the Fermionic Fock space with values in $T_{\gamma_{1/2}}(M)$ of the flat Brownian bridge. But if γ_0, γ_1, $\gamma_{1/2}$ are close enough to M_i, we can introduce the parallel transport along the unique geodesic joining M_i to γ_0, the parallel transport from M_i to γ_1 along the unique geodesic joining these two points and the parallel transport from M_i to $\gamma_{1/2}$ along the unique the geodesic joining these two points. In this case, the fiber is isomorphic to $\Lambda_L(M_i) \wedge \Lambda_{L'}(M_i) \wedge \Lambda_{M_i}(H)$. In the other case, we choose as fiber the original $\Lambda T_L(\gamma_0) \wedge \Lambda T_{L'}(\gamma_1) \wedge \Lambda_{\gamma_{1/2}}(H)$ and the space of L^2 section in $\gamma_0, \gamma_1, \gamma_{1/2}$ of this Hilbert bundle over $L \times L' \times M$. We have a random variable in this space of sections, when we are far from the intersection points. In this case, if $\sigma \in \Lambda$, $E_\epsilon[|\sigma|^2] \to 0$ and therefore $\sigma \to 0$ in law, when we are far from the intersection points, when we do not perform any Bismut's dilatation.

We have the following theorem, which justifies the introduction of the limit model.

Theorem 2.1. *For any fixed σ element of Λ where Bismut's dilatations are defined, we have, in law:*

$$B_\epsilon \sigma \to \sigma_l, \tag{2.24}$$

$$\mathrm{d}_{\epsilon\mathrm{rWZW}} B_\epsilon \sigma \to \mathrm{d}_{\mathrm{lWZW}} \sigma_l, \qquad \mathrm{d}^*_{\epsilon\mathrm{rWZW}} B_\epsilon \sigma \to \mathrm{d}^*_{lWZW} \sigma_l. \tag{2.25}$$

Proof. We work in normal charts in M_i. We perform in M_i the rescaling $x \to \epsilon x$ in the direction of T_L and $y \to \epsilon y$ in the direction of $T_{L'}$. Let us recall that $T_L \oplus T_{L'} = T_M$ in M_i. This has the effect of canceling the ϵ^{-d} which occurs from the asymptotic expansion of the heat kernel near the diagonal (see [Bi4] for analoguous considerations):

$$p_{\epsilon^2}(x, y) = \frac{C}{\epsilon^d} \exp\left[-\frac{d^2(x, y)}{2\epsilon^2}\right] \left(\sum a_i(x, y)\epsilon^{2i} + \mathrm{o}(\epsilon^N)\right) \tag{2.26}$$

(see [L1]). Moreover in a system of exponential charts near M_i

$$\gamma_s = \epsilon x(1 - s) + s\epsilon y + \epsilon B_{s,\mathrm{flat}} + \epsilon \nu_2 s \tag{2.27}$$

with a greater probability (see [Bi4]). Let us explain the role of ν_2. For this, let us introduce the canonical horizontal vector fields X_i over the Riemannian manifold. Let us study the equation over the frame bundle

$$\mathrm{d}u_s = \sum X_i(u_s)(\epsilon(-x + y) + \epsilon\, \mathrm{d}B_{s,\mathrm{flat}} + \epsilon \nu_2 s). \tag{2.28}$$

Let us denote by πu_s the canonical projection of u_s over the Riemannian manifold. $\pi u_0 = M_i + \epsilon x$. We choose ν_2 such that $\pi u_1 = M_i + \epsilon y$. It is asymptotically possible with a greater probability (see [Bi2,Bi3]), and the error term cancels when $\epsilon \to 0$.

(a) Let us show that in law $B_\epsilon \sigma \to \sigma_l$. We have

$$1/\epsilon(f(\gamma_{t_i}) - f(\Pi\gamma_0)) = 1/\epsilon(f(\gamma_{t_i}) - f(\gamma_0)) + 1/\epsilon(f(\gamma_0) - f(\Pi\gamma_0)). \tag{2.29}$$

This tends in law to

$$\begin{aligned}&\int_0^{t_i} \langle \mathrm{d}f(M_i), -x + y + \mathrm{d}B_{s,\mathrm{flat}}\rangle + \langle \mathrm{d}f(M_i), x\rangle \\ &\quad = \langle \mathrm{d}f(M_i), (1 - t_i)x\rangle + \langle \mathrm{d}f(M_i), t_i y\rangle + \langle \mathrm{d}f(M_i), B_{t_i,\mathrm{flat}}\rangle.\end{aligned} \tag{2.30}$$

After choosing over the M_i the law given before, we deduce the first point.

(b) Let us show that $\mathrm{d}_{\epsilon\mathrm{rWZW}} B_\epsilon \sigma \to \mathrm{d}_{\mathrm{lWZW}} \sigma_l$ in law. We consider derivatives along the vector field $\epsilon X_{n,i}, \epsilon Y_i, \epsilon Y_i'$. Therefore at the limit no derivatives of the σ_l appear. It remains to consider the derivatives of $B_\epsilon F_l$. Let us study the derivative of $(f(\gamma_{t_i}) - f(\Pi(\gamma_0)))/\epsilon$ over $\epsilon X_{n,i}, \epsilon Y_i, \epsilon Y_i'$. It is

$$\begin{aligned}&\langle \mathrm{d}f(\gamma_{t_i}), X_{n,i,t_i}\rangle - \langle \mathrm{d}f(\gamma_0), X_{n,i,0}\rangle \\ &\quad + \langle \mathrm{d}f(\gamma_0), X_{n,i,0}\rangle - \langle \mathrm{d}f(\Pi\gamma_0), X_{n,i,0}\rangle.\end{aligned} \tag{2.31}$$

But, $X_{n,i,0} = 0$. $f(\Pi\gamma_0)$ is constant, and therefore its derivative equals 0. So at the limit, we recognize the derivative of $\langle \mathrm{d}f(M_i), B_{t_i,\mathrm{flat}}\rangle$ over the flat Brownian bridge vector field $H_n(e_i)$: $H_n = C(\cos(2\pi ns) - 1)/n$ or $H_n = C(\sin(2\pi ns))/n)$ if $n > 0$ or if $n < 0$. The derivatives over $H_n(e_i)$ of $\langle \mathrm{d}f(M_i), (1 - t_i)x\rangle$ or $\langle \mathrm{d}f(M_i), t_i y\rangle$ are equal to 0.

Let us study the derivative following ϵY_i. It is

$$\langle \mathrm{d}f(\gamma_{t_i}), \tau_{t_i}(1 - t_i)e_i\rangle. \tag{2.32}$$

At the limit, it is $\langle \mathrm{d}f(M_i), (1-t_i)X\rangle$, which is the derivative of $\langle \mathrm{d}f(M_i), \gamma_{t_i,\mathrm{flat}}\rangle$ along the derivatives of the vector field $(1-t_i)X$.

We have the same computations for the derivatives over $\epsilon Y_i'$.

The better originality with respect to [LR] arises from the Wess–Zumino–Witten term. It is

$$\sum \frac{\langle \mathrm{d}F, \epsilon X_{n,i}\rangle}{\epsilon^2} X_{n,i} \wedge + \sum \frac{\langle \mathrm{d}F, \epsilon Y_i\rangle}{\epsilon^2} Y_i \wedge + \sum \frac{\langle \mathrm{d}F, \epsilon Y_i'\rangle}{\epsilon^2} Y_i' \wedge . \tag{2.33}$$

But the family of $\langle \mathrm{d}F, \epsilon X_{n,i}\rangle/\epsilon^2$, $\langle \mathrm{d}F, \epsilon Y_i\rangle/\epsilon^2$ and $\langle \mathrm{d}F, \epsilon Y_i'\rangle/\epsilon^2$ tends in law to the family of $\int_0^1 \omega(\mathrm{d}\gamma_{s,\mathrm{flat}}, X_{n,i,s})$, $\int_0^1 \omega(\mathrm{d}\gamma_{s,\mathrm{flat}}, Y_{i,s})$ and $\int_0^1 \omega(\mathrm{d}\gamma_{s,\mathrm{flat}}, Y'_{i,s})$, each term being bounded in L^2 by C/n in the first expression. This shows us that

$$\mathrm{d}F \wedge B_\epsilon \sigma \to \mathrm{d}F_l \wedge \sigma_l. \tag{2.34}$$

(c) Let us now study $\mathrm{d}^*_{\epsilon r\mathrm{WZW}} B_\epsilon\sigma$. It is a finite sum. The term in $-i_{X_{n,i}}\nabla_{\epsilon X_{n,i}} B_\epsilon\sigma$, $-i_{Y_i}\nabla_{\epsilon Y_i} B_\epsilon\sigma$, $-i_{Y_i'}\nabla_{\epsilon Y_i'} B_\epsilon\sigma$ are treated as in (b). Moreover,

$$\operatorname{div} \epsilon X_{n,i} = \frac{\epsilon \int_0^1 \langle DX_{n,i}, \delta\gamma_s\rangle}{\epsilon^2} + \epsilon\frac{\epsilon^2}{\epsilon^2}\int_0^1 \langle S_{X_{n,i}}, \delta\gamma_s\rangle + \text{counterterm}. \tag{2.35}$$

Therefore the family

$$\operatorname{div} \epsilon X_{n,i} \to \int_0^1 \langle X'_{n,i}, \delta\gamma_{s,\mathrm{flat}}\rangle \tag{2.36}$$

in law. The main difference with [LR] lies in $\operatorname{div}\epsilon Y_i$ and $\operatorname{div}\epsilon Y_i'$. But

$$\operatorname{div} \epsilon Y_i = \epsilon \operatorname{div} e_i - \frac{\epsilon}{\epsilon^2}\int_0^1 \langle \tau_s e_i, \delta\gamma_s\rangle + \frac{\epsilon^2}{\epsilon^2}\int_0^1 \langle S_{\epsilon Y_i}, \delta\gamma_s\rangle, \tag{2.37}$$

which tends in law to $\langle e_i, x-y\rangle$, the divergence of $e_i \in T_L$ over the limit model. We have a similar type of computations for $\operatorname{div}\epsilon Y_i'$.

Let us study the Wess–Zumino–Witten term. It is

$$\sum \frac{\langle \mathrm{d}F, \epsilon X_{n,i}\rangle}{\epsilon^2} i_{X_{n,i}} + \sum \frac{\langle \mathrm{d}F, \epsilon Y_i\rangle}{\epsilon^2} i_{Y_i} + \sum \frac{\langle \mathrm{d}F, \epsilon Y_i'\rangle}{\epsilon^2} i_{Y_i'}. \tag{2.38}$$

By considerations same as those for the Wess–Zumino–Witten term for $\mathrm{d}_{\epsilon r\mathrm{WZW}}$, we see that it tends in law to $i_{\mathrm{d}F_l}\sigma_l$.

Remark. We separate, in order to give a nicer exposure, the convergence in law of the different pieces of the considered series. It is not completely correct, but the convergence in law of the global expression is ensured by Bismut's procedure (2.28).

Let us now introduce the limit Wess–Zumino–Witten Laplacian; in the case of a general interacting term, it was extensively studied by Arai [Ar]:

$$\Delta_{\mathrm{lWZW}} = (\mathrm{d}_l + \mathrm{d}F_l + \mathrm{d}_l^* + i_{\mathrm{d}F_l})^2. \tag{2.39}$$

Theorem 2.2. *Let us suppose that* $E[\exp[2|F_l|]] < \infty$. Δ_{lWZW} *is a harmonic oscillator which has* $\exp[-F_l]$ *as unique ground-state.*

Remark. We will do away the convention that $\exp[-F_l]$ is in L^2. If it is not the case, we can take $\exp[-\lambda F_l]$.

Proof. Let us recall that $\mathrm{d} + \mathrm{d}F_l\wedge$ is complex because it is equal to $\exp[-F_l]\,\mathrm{d}\exp[F_l]$. Therefore,

$$\begin{aligned}\Delta_{\mathrm{lWZW}} &= (\mathrm{d} + \mathrm{d}F_l\wedge)(\mathrm{d}^* + i_{\mathrm{d}F_l}) + (\mathrm{d}^* + i_{\mathrm{d}F_l})(\mathrm{d} + \mathrm{d}F_l\wedge)\\ &= \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d} + \mathrm{d}F_l i_{\mathrm{d}F_l} + i_{\mathrm{d}F_l}\,\mathrm{d}F_l + \mathrm{d}i_{\mathrm{d}F_l} + \mathrm{d}F_l\,\mathrm{d}^* + i_{\mathrm{d}F_l}\,\mathrm{d} + \mathrm{d}^*\,\mathrm{d}F_l\end{aligned} \tag{2.40}$$

Let us introduce the Bosonic Number operator N_{B} and the Fermionic number operator N_{F}. We have [Sh]

$$\mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d} = N_{\mathrm{B}} + N_{\mathrm{F}}. \tag{2.41}$$

Moreover, clearly,

$$\mathrm{d}F i_{\mathrm{d}F} + i_{\mathrm{d}F}\,\mathrm{d}F = \|\,\mathrm{d}F\,\|^2. \tag{2.42}$$

Let us introduce an orthonormal basis x_i of the limit abstract Wiener space. We have

$$\mathrm{d} = \sum \frac{\partial}{\partial x_i}\,\mathrm{d}x_i, \tag{2.43}$$

$$\mathrm{d}^* = -\sum \frac{\partial}{\partial x_i} i_{\mathrm{d}x_i} + x_i i_{\mathrm{d}x_i}. \tag{2.44}$$

This shows us that:

$$\mathrm{d}i_{\mathrm{d}F_l} = \sum \frac{\partial}{\partial x_i}\,\mathrm{d}x_i \left(\sum \frac{\partial F}{\partial x_j} i_{\mathrm{d}x_j}\right) = \sum \frac{\partial^2 F}{\partial x_i \partial x_j}\,\mathrm{d}x_i i_{\mathrm{d}x_j} + \sum \frac{\partial F}{\partial x_j}\,\mathrm{d}x_i i_{\mathrm{d}x_j} \frac{\partial}{\partial x_i}. \tag{2.45}$$

Moreover,

$$\begin{aligned}\mathrm{d}F_l\,\mathrm{d}^* &= \sum \frac{\partial F}{\partial x_j}\,\mathrm{d}x_j \left(-\sum \frac{\partial}{\partial x_i} i_{\mathrm{d}x_i} + x_i i_{\mathrm{d}x_i}\right)\\ &= \sum \frac{\partial F}{\partial x_j}\,\mathrm{d}x_j x_i i_{\mathrm{d}x_i} - \sum \frac{\partial F}{\partial x_j}\,\mathrm{d}x_j i_{\mathrm{d}x_i} \frac{\partial}{\partial x_i}\end{aligned} \tag{2.46}$$

and

$$i_{\mathrm{d}F_l}\,\mathrm{d} = \sum \frac{\partial F}{\partial x_j} i_{\mathrm{d}x_j}\,\mathrm{d}x_i \frac{\partial}{\partial x_i}. \tag{2.47}$$

Finally

$$\mathrm{d}^*\,\mathrm{d}F_l = N_{\mathrm{B}} F_l - \sum_{i\neq j} \frac{\partial^2 F}{\partial x_i \partial x_j} i_{\mathrm{d}x_i}\,\mathrm{d}x_j + \sum \frac{\partial^2 F}{\partial x_i^2}\,\mathrm{d}x_i i_{\mathrm{d}x_i}$$
$$+ \sum_{i\neq j} \frac{\partial F}{\partial x_j} x_i i_{\mathrm{d}x_i}\,\mathrm{d}x_j - \sum \frac{\partial F}{\partial x_j} i_{\mathrm{d}x_i}\,\mathrm{d}x_j \frac{\partial}{\partial x_i} - \sum \frac{\partial F}{\partial x_i} x_i\,\mathrm{d}x_i i_{\mathrm{d}x_i}. \tag{2.48}$$

If we sum the differential terms in (2.45) and (2.46), since $\mathrm{d}x_i i_{\mathrm{d}x_j} + i_{\mathrm{d}x_j}\,\mathrm{d}x_i = \delta_{i,j}$, we find $2\sum(\partial F/\partial x_i)(\partial/\partial x_i)$. But this cancels, if we sum the differential terms in (2.46) and (2.48). This shows us that $\mathrm{d} i_{\mathrm{d}F_l} + \mathrm{d}F_l\,\mathrm{d}^* + i_{\mathrm{d}F_l}\,\mathrm{d} + \mathrm{d}^*\,\mathrm{d}F_l = C_l$ is a tensorial operator. Moreover,

$$C_l = N_B F_l + 2\sum_{i\neq j} \frac{\partial^2 F}{\partial x_i \partial x_j}\,\mathrm{d}x_i i_{\mathrm{d}x_j} + 2\sum \frac{\partial^2 F}{\partial x_i^2}\,\mathrm{d}x_i i_{\mathrm{d}x_i}. \tag{2.49}$$

Let us recall that for $a_{i,j}$ symmetric Hilbert–Schmidt, we have

$$F = \sum a_{i,j}(x_i x_j - \delta_{i,j}) + C. \tag{2.50}$$

We deduce that

$$C_l = N_{\mathrm{B}} F_l + 4\sum_{i\neq j} a_{i,j}\,\mathrm{d}x_i i_{dx_j} + 4\sum a_{i,i}\,\mathrm{d}x_i i_{\mathrm{d}x_i}. \tag{2.51}$$

Let $\sigma_k = \sum \sigma_{k,i}\,\mathrm{d}x_i$. $\sum \sigma_{k,i}^2 = 1$. $\sigma_1 \wedge \cdots \wedge \sigma_k$ is in the domain of C_l.

For $k = 1$, it is clear. Namely $\sum a_{i,i}\sigma_{1,i}\,\mathrm{d}x_i$ converges because $\sum a_{i,i}^2 < \infty$, by the Cauchy–Schwartz inequality. Moreover, $\sum_{i\neq j} a_{i,j}\,\mathrm{d}x_i \sigma_{1,j}$ has a norm bounded by $\sum(\sum_j a_{i,j}\sigma_{1,j})^2 \leq \sum(\sum_j a_{i,j}^2)(\sum \sigma_{1,j}^2)$ which is finite because $a_{i,j}$ is Hilbert–Schmidt.

It is enough to study the case $k = 2$, because $\mathrm{d}x_i i_{\mathrm{d}x_j}$ can act only over two elements of the wedge product. The disturbing case is when $i_{\mathrm{d}x_j}$ acts over the first one and $\mathrm{d}x_i$ over the second one. We get

$$\sum_{i\neq j} a_{i,j}\,\mathrm{d}x_i i_{\mathrm{d}x_j} \left(\sum_j \sigma_{1,j}\,\mathrm{d}x_j \wedge \sum_{j'} \sigma_{2,j'}\,\mathrm{d}x_{j'} \right) = \sum_{i\neq j} a_{i,j}\,\mathrm{d}x_i \sigma_{1,j} \wedge \sum \sigma_{2,j'}\,\mathrm{d}x_{j'}. \tag{2.52}$$

Its norm is

$$\sum_{i,j,j'} \left(\sum a_{i,j}\sigma_{1,j}\right)^2 \sigma_{2,j'}^2 \leq \sum_{i,j,j',j''} a_{i,j}^2 \sigma_{1,j'}^2 \sigma_{2,j''}^2 < \infty, \tag{2.53}$$

Let us do the same hypothesis over $\mathrm{d}x_i i_{\mathrm{d}x_i}$. We get

$$\sum a_{i,i}\,\mathrm{d}x_i i_{\mathrm{d}x_i}\left(\sum_j \sigma_{1,j}\,\mathrm{d}x_j \wedge \sigma_{2,j'}\,\mathrm{d}x_{j'}\right) = \sum a_{i,i}\,\mathrm{d}x_i \sigma_{1,i} \wedge \sum_{j'} \sigma_{2,j'}\,\mathrm{d}x_{j'} + \text{term}, \tag{2.54}$$

which belongs in L^2.

Let us diagonalize $a_{i,j}$. We find

$$C_l = \sum 2\lambda_i (x_i^2 - 1) + \sum 4\lambda_i\,\mathrm{d}x_i i_{\mathrm{d}x_i} \tag{2.55}$$

and

$$N_{\mathrm{B}} = -\sum \frac{\partial^2}{\partial x_i^2} + x_i \frac{\partial}{\partial x_i} \tag{2.56}$$

and finally

$$N_{\mathrm{F}} = \sum \mathrm{d}_{x_i} i_{\mathrm{d}x_i}. \tag{2.57}$$

Δ_{lWZW} can be split into a series of commuting operators of the shape

$$-\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + 2\lambda(x^2 - 1) + 4\lambda\,\mathrm{d}_x i_{\mathrm{d}x} + \mathrm{d}x i_{\mathrm{d}x} + 4\lambda^2 x^2 = \Delta_l. \tag{2.58}$$

In order to diagonalize Δ_l, let us try to get a harmonic oscillator.

First, we use the constatation that the scalar Ornstein–Uhlenbeck operator $-\partial^2/\partial x^2 + x\partial/\partial x$ is a harmonic oscillator when we use the transformation $f \to \exp[x^2/4]f$. We have therefore to diagonalize for the Lebesgue measure the operator

$$-\frac{\partial^2}{\partial x^2} + x^2\left(4\lambda^2 + 2\lambda + \tfrac{1}{4}\right) - 2\lambda i_{\mathrm{d}x}\,\mathrm{d}x + 2\lambda\,\mathrm{d}x i_{\mathrm{d}x} - \tfrac{1}{2} + \mathrm{d}x i_{dx} = \Delta_l. \tag{2.59}$$

We recognize modulo the number operator $\mathrm{d}x i_{\mathrm{d}x}$ a supersymmetric harmonic oscillator. The eigenvalues of the first one are $(2k+1)(|2\lambda + \frac{1}{2}|)$ [DW]. The second operator

$$-2\lambda i_{\mathrm{d}x}\,\mathrm{d}x + 2\lambda\,\mathrm{d}x i_{\mathrm{d}x} - \tfrac{1}{2} + \mathrm{d}x i_{\mathrm{d}x} \tag{2.60}$$

has eigenvalues $\pm(2\lambda + \frac{1}{2})$, whether we have a fermion or not.

But we have supposed that

$$E[\exp[2|F_l|]] < \infty. \tag{2.61}$$

Therefore, $|\lambda| < \frac{1}{4}$ and $2\lambda + 1 > 0$. Therefore there is only one element in the kernel, and it is when we do not have any fermion. This proves the theorem. □

Theorem 2.3. *If $\sigma \in \Lambda$, Bismut's dilatation are defined, and we have, in law,*

$$B_\epsilon \sigma \to \sigma_l \tag{2.62}$$

and

$$\Delta_{\epsilon,\mathrm{rWZW}} B_\epsilon \sigma \to \Delta_{\mathrm{lWZW}} \sigma_l. \tag{2.63}$$

Proof. We follow the line of Theorem 1.4.

(a) Let us recall that we take $\epsilon X_{n,i}$ or ϵY_i or $\epsilon Y'_i$. So, if we take at least one derivative of σ_l, this tends to 0. So we have to consider the limit of two derivatives of $B_\epsilon F_l$. We use (2.31) and (2.32) in order to show that this tends to two derivatives of the limit functional without the parallel transport. Therefore $d_{\epsilon,r} d_{\epsilon,r} B_\epsilon \sigma \to d_l d_l \sigma_l = 0$ in law.

(b) $\mathrm{d}^*_{\epsilon r}\,\mathrm{d}^*_{\epsilon r} B_\epsilon \sigma$ is a finite sum. We use (2.35) and (2.37) in order to study

$$\langle \mathrm{d}(\mathrm{div}(\epsilon X_{n,i})), \epsilon X_{n',i'} \rangle \tag{2.64}$$

and the other analoguous formulas. We see that it tends in law to

$$\int_0^1 \langle X'_{n,i}, X'_{n',i'} \rangle, \tag{2.65}$$

which is the derivative of $\int_0^1 \langle X'_{n,i}, \delta\gamma_{s,\mathrm{flat}} \rangle$ along the flat vector field $X_{n',i'}$. We have the analoguous formulas for the other divergences and the other derivatives. This shows us that $\mathrm{d}^*_{\epsilon r}\,\mathrm{d}^*_{\epsilon r} B_\epsilon \sigma$ tends in law to $\mathrm{d}^*_l\,\mathrm{d}^*_l \sigma_l = 0$, by using analoguous considerations as in (a).

(c) $\mathrm{d}^*_{\epsilon r}\,\mathrm{d}_{\epsilon r} B_\epsilon \sigma$ is more complicated. The most complicated term to treat is

$$\begin{aligned} &\sum -\nabla_{\epsilon X_{n,i}} i_{X_{n,i}} \wedge X_{n,i} \nabla_{\epsilon X_{n,i}} B_\epsilon \sigma \\ &\quad + \sum \mathrm{div}\epsilon X_{n,i} i_{X_{n,i}} \wedge X_{n,i} \nabla_{\epsilon X_{n,i}} B_\epsilon \sigma = A(\epsilon) + B(\epsilon). \end{aligned} \tag{2.66}$$

In $A(\epsilon)$, the terms which remain when $\epsilon \to 0$ are the terms when we take two derivatives of $B_\epsilon F_l$. It can be treated as in (a). $A(\epsilon)$ tends in law to $\sum -\nabla_{X_{n,i}} i_{X_{n,i}} \wedge X_{n,i} \nabla_{X_{n,i}} \sigma_l$.

Let us study $B(\epsilon)$. We recognize an Itô integral by taking one derivative of σ_l tends to zero because $\nabla_{\epsilon X_{n,i}} \sigma_l$ is in ϵ/n. So we have to take the derivative of $B_\epsilon F_l$. We recognize a sum of products of terms which by (a) converges in law to a non-anticipative Itô integral, which converges by (2.35).

This shows us that $\mathrm{d}^*_{\epsilon r}\,\mathrm{d}_{\epsilon r} B_\epsilon \sigma$ converges in law to $\mathrm{d}^*_l\,\mathrm{d}_l \sigma_l$.

(d) Let us consider the case of $\mathrm{d}_{\epsilon r} \wedge \mathrm{d}F/\epsilon^2$. Let us recall that $\mathrm{d}F/\epsilon^2$ is equal to

$$\sum \int_0^1 \frac{\omega(\mathrm{d}\gamma_s, X_{n,i})}{\epsilon} \wedge X_{n,i} + \sum \int_0^1 \frac{\omega(\mathrm{d}\gamma_s, Y_i)}{\epsilon} \wedge Y_i + \sum \int_0^1 \frac{\omega(\mathrm{d}\gamma_s, Y'_i)}{\epsilon} \wedge Y'_i. \tag{2.67}$$

If we derive $B_\epsilon \sigma$, we create a series of $X_n \wedge X_m$ multiplied by a term in $C/[(|n|+1)(|m|+1)]$, and C tends to zero if we take a derivative of σ_i. So if we take derivative of $B_\epsilon \sigma$, we have to take only one derivative of $B_\epsilon F_l$, and since in law $\int_0^1 \omega(\mathrm{d}\gamma_s, X_{n,i})/\epsilon \to \int_0^1 \omega(\mathrm{d}\gamma_{s,\mathrm{flat}}, X_{n,i})$ and is bounded by C/n, we have, if we take the derivative of $B_\epsilon \sigma$, the limit behavior of $\mathrm{d}_l\ \mathrm{d}F_l \wedge \sigma_l$ when we take the derivative of σ_l.

The difficult term is when we take the derivative of $\mathrm{d}F/\epsilon^2$. The most complicated term is as in (e), Theorem 1.4,

$$\sum \int_0^1 \omega(\gamma_s)(DX_{m,i}, X_{n,j}) X_{m,i} \wedge X_{n,j}. \tag{2.68}$$

This is cancellation in this term, and it tends to zero when $\epsilon \to 0$, because in the integration by part (1.47), we get $O(1/(|n|+1)(|m|+1))$ which tends uniformly to zero when $\epsilon \to 0$.

(e) In a simpler way, we see that $\mathrm{d}F/\epsilon^2 \wedge \mathrm{d}_{\epsilon,r} B_\epsilon \sigma$ tends in law to $\mathrm{d}F_l \wedge \mathrm{d}_l \sigma_l$.

(f) $\mathrm{d}F/\epsilon^2 . i_{\mathrm{d}F/\epsilon^2} + i_{\mathrm{d}F/\epsilon^2} . \mathrm{d}F/\epsilon^2$ is equal to

$$\sum \left(\int_0^1 \frac{\omega(\mathrm{d}\gamma_s, X_{n,i})}{\epsilon} \right)^2 + \sum \left(\int_0^1 \frac{\omega(\mathrm{d}\gamma_s, Y_i)}{\epsilon} \right)^2 + \sum \left(\int_0^1 \frac{\omega(\mathrm{d}\gamma_s, Y_i')}{\epsilon} \right)^2, \tag{2.69}$$

which tends in law to $\| \mathrm{d}F_l \|^2$.

(g) $\mathrm{d}F/\epsilon^2 \wedge \mathrm{d}^*_{\epsilon r} B_\epsilon \sigma$ tends in law to $\mathrm{d}F_l \wedge \mathrm{d}^*_l \sigma_l$ because $\mathrm{d}^*_{\epsilon r} B_\epsilon \sigma$ is a finite sum.

(h) It is the same for $\mathrm{d}^*_{\epsilon r} i_{\mathrm{d}F/\epsilon^2} B_\epsilon \sigma$.

(i) $\mathrm{d}_{\epsilon r} i_{\mathrm{d}F/\epsilon^2}$ does not cause any difficulty. Namely $\langle \mathrm{d} \int_o^1 \omega(\mathrm{d}\gamma_s, X_{n,i})/\epsilon, \epsilon X_{n',i'} \rangle$ behaves as $\int_0^1 \omega(X'_{n',i'}, X_{n,i})$ at the limit, the divergence being cancelled by the integration by part (1.47). This last expression is the derivative along $X_{n',i'}$ of $\int_0^1 \omega(\mathrm{d}\gamma_{s,\mathrm{flat}}, X_{n,i})$.

In $i_{\mathrm{d}F/\epsilon^2} \wedge \mathrm{d}_{\epsilon r}$, we create an infinite sum of $X_{n,i} \wedge X_{n',i'}$; each component is bounded by $C/[(|n|+1)(|n'|+1)]$; the derivatives of σ_l are cancelling, and the most complicated term to consider is $\int_0^1 \omega(\mathrm{d}\gamma_s, X_{n,i})/\epsilon \langle \mathrm{d}B_\epsilon F_l, X_{n',i'} \rangle$. This tends in law to $i_{\mathrm{d}F_l} \wedge \mathrm{d}_l$.

(j) The last term to study is $\mathrm{d}^*_{\epsilon r} \mathrm{d}F/\epsilon^2$. The most boring term is

$$-\sum \left\langle \mathrm{d} \int_0^1 \omega(\mathrm{d}\gamma_s, X_{n,i}), X_{n,i} \right\rangle + \sum \int_0^1 \omega(\mathrm{d}\gamma_s, X_{n,i}) \operatorname{div} X_{n,i} = \sum \operatorname{div} \tilde{X}_i, \tag{2.70}$$

where $\tilde{X}_i$ is the vector field over the based loop space given by

$$D\tilde{X}_{i,s} = \tau_s \left(-\int_0^s \omega(\mathrm{d}\gamma_u, \tau_u \tau_{1/2}^{-1} e_i) + \int_0^1 \omega(\mathrm{d}\gamma_u, \tau_u \tau_{1/2}^{-1} e_i) + C \right) \tau_{1/2}^{-1} e_i. \tag{2.71}$$

C is introduced in order to get a vector over the based loop such that its average is equal to zero. We use the fact that $s \to \int_0^s \omega(\mathrm{d}\gamma_u, \tau_u \tau_{1/2}^{-1} e_i)$ is adapted modulo-the term in $\tau_{1/2}$ whose derivative tends to zero. We deduce from (2.35), that in law

$$\operatorname{div} \left(-\int_0^s \omega(\mathrm{d}\gamma_u, \tau_u \tau_{1/2}^{-1} e_i) \tau_{1/2}^{-1} e_i \right) \to -\int_{0<u<s<1} \omega(\mathrm{d}\gamma_{u,\mathrm{flat}}, e_i) \langle e_i, \mathrm{d}B_{s,\mathrm{flat}} \rangle \tag{2.72}$$

over the based path space. The other terms in time s are constant. In particular in law, over the based path space

$$\operatorname{div}\int_0^1 \omega(\mathrm{d}\gamma_u, \tau_u\tau_{1/2}^{-1}e_i) \to \int_0^1 \omega(\mathrm{d}\gamma_{u,\mathrm{flat}}, e_i)\int_0^1 \langle e_i, \mathrm{d}B_{s,\mathrm{flat}}\rangle - \int_0^1 \omega(e_i\,\mathrm{d}s, e_i). \tag{2.73}$$

The last term vanishes because ω is antisymmetric. For the computation of the divergence of C in (2.70), we operate as in the first case. Let us recall that we want in fact to compute this divergence over the pinned Brownian bridge. We proceed as in (1.49) and in (1.50). We use the fact that ω is antisymmetric, and we see that in law over the Brownian bridge

$$\operatorname{div}\tilde{X}_i \to \int_{0<s<u<1} \langle e_i, \mathrm{d}B_{s,\mathrm{flat}}\rangle\omega(\mathrm{d}\gamma_{u,\mathrm{flat}}, e_i). \tag{2.74}$$

3. Cohomology groups

$\mathrm{d}_{\mathrm{rWZW}}$ does not define a complex. We can define a complex following the lines of Léandre [L5].

Let σ be an n-form over $P(L, L')$. In local coordinates over L and L', we can write σ as

$$\sigma = \sum \sigma_{J,J'} \wedge \mathrm{d}x_J \wedge \mathrm{d}x_{J'}, \tag{3.1}$$

where $\mathrm{d}x_J$ is a set of forms over L and $\mathrm{d}x_{J'}$ is a set of forms over L' we get by taking the different possible wedge products of a local smooth orthonormal basis of T_L and of $T_{L'}$. $\sigma_{J,J'}$ appears as a form over the tangent space of the pinned Brownian bridge going from $x \in L$ to $y \in L'$.

Therefore $\sigma_{J,J'}$ is given by kernels

$$\sigma_{J,J'} = \sigma_{J,J'}(s_1, \dots, s_l), \tag{3.2}$$

where $\sigma_{J,J'}(s_1, \dots, s_l)$ is a l-cotensor over $T_M(\gamma_0)$ and such that

$$\int_0^1 \sigma_{J,J'}(s_1, \dots, s_l)\mathrm{d}s_i = 0 \tag{3.3}$$

because we operate for such kernels over the tangent space of the pinned Brownian motion.

We can take the covariant derivative of $\sigma_{J,J'}$ as a l-cotensor over $T_M(\gamma_0)$ by taking the pullback ∇_0 of the Levi-Civita connection over T_M by the evaluation map $\gamma \to \gamma_0$.

$\nabla_0^k\sigma_{J,J'}$ is given by the kernels $\sigma_{J,J'}(s_1, \dots, s_l; t_1, \dots, t_{k'})$ $k \le k$. We say that $\sigma_{J,J'}$ belongs to the Sobolev space $N_{k,p}$ in the sense of Nualart–Pardoux if

$$\begin{aligned}&\|\sigma_{J,J'}(s_1, \dots, s_l; t_1, \dots, t_{k'}) - \sigma_{J,J'}(s'_1, \dots, s'_l; t'_1, \dots, t'_{k'})\|_{L^p}\\ &\quad\le C(p,k)\left(\sum\sqrt{|s_i - s'_i|} + \sum\sqrt{|t_j - t'_j|}\right)\end{aligned} \tag{3.4}$$

over each components of the diagonals and if

$$\sup \|\sigma_{J,J'}(s_1, \ldots, s_l; t_1, \ldots, t_l)\|_{L^p} = C'(p,k) < \infty. \tag{3.5}$$

This works modulo a partition of unity over L and L' $O(L, L')$. We set as Nualart–Pardoux Sobolev norms of an n-form:

$$\|\sigma\|_{p,k} = \frac{2^{np}}{(n-p)!n!} \sum_{k' \leq k} \sum_{O(L,L')} \sum_{J,J'} \{C(p,k')(\sigma_{J,J'}) + C'(p,k')(\sigma_{J,J'})\}. \tag{3.6}$$

We get equivalent norms when we change the system of partition of unity and the system of local orthonormal basis of the tangent space of L or of L'.

If we consider a series of n forms $\sigma = \sum \sigma_n$, we take as Sobolev Nualart–Pardoux norm of σ the expression:

$$\|\sigma\|_{p,k} = \sum \|\sigma_n\|_{p,k}. \tag{3.7}$$

Let us recall that $\nabla_0^k \tau_1$ is a cotensor which checks the Nualart–Pardoux conditions. Namely by the Bismut formula:

$$\nabla_{0,X} \tau_1 = \tau_1 \int_0^1 \tau_s^{-1} R(\mathrm{d}\gamma_s, X_s)\tau_s \tag{3.8}$$

and its kernel checks the Nualart–Pardoux conditions. The kernel of $\nabla_0^k \tau_1$ over the open Brownian motion checks the Nualart–Pardoux conditions, because, by iterating (3.8), and using the fact that a product of iterated integrals is still a sum of iterated integrals, the kernel of $\nabla_0^k \tau_1$ is given by iterated integrals with frozen terms, which check the Nualart–Pardoux conditions over the open Brownian motion. By averaging, we deduce that they still satisfy the Nualart–Pardoux conditions for the pinned Brownian motion.

We say that a form is smooth in the Nualart–Pardoux sense, if it belongs to all the Nualart–Pardoux spaces.

Let us operate in γ_1 instead of γ_0. The kernel of the form is transformed in $\sigma(s_1, \ldots, s_n)$ $\tau_1^{-1} \ldots \tau_1^{-1}$, and we have to consider the covariant derivatives ∇_1^k of the transformed kernels. But,

$$\nabla_1 \tau_1 H_1 = (\nabla_0 \tau_1) H_1 + \tau_1 \nabla_0 H_1. \tag{3.9}$$

We deduce that we get the same space of forms smooth in the Nualart–Pardoux sense if we interchange the role of L and of L', because $\nabla_0^k \tau_1$ satisfy the Nualart–Pardoux conditions, and that the set of Nualart–Pardoux Sobolev norms are equivalent when we interchange the role of L and L'.

Let us recall that the exterior derivative of an n-form σ is defined as follows:

$$\begin{aligned} \mathrm{d}\sigma(X_1, \ldots, X_{n+1}) = &\sum (-1)^{i-1} \langle \mathrm{d}\sigma(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}), X_i \rangle \\ &+ \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{n+1}), \end{aligned} \tag{3.10}$$

where $\hat{}$ denotes the omission operator and the X_i are vector fields.

From (3.8), we deduce that the problem of defining $\mathrm{d}\sigma$ is strongly related to the problem of defining an anticipative Stratonovitch integral over the pinned brownian motion. We can use the analoguous in this situation of Lemma I.2 of Léandre [L6] in order to deduce that:

Theorem 3.1. *The exterior derivative is continuous over the space $N.P_\infty$ of forms smooth in the Nualart–Pardoux sense.*

Moreover, the exterior product is continuous over the space of forms smooth in the Nualart–Pardoux sense (see [L5, Theorem I.2]). We can compute the kernel of $\mathrm{d}F = \int_0^1 \omega(\mathrm{d}\gamma_s, X_s)$. It is $\int_s^1 \omega(\mathrm{d}\gamma_u, \tau_u) - \int_0^1 \mathrm{d}s \int_s^1 \omega(\mathrm{d}\gamma_u, \tau_u)$. The $\mathrm{d}x_j$ part is $\int_0^1 \omega(\mathrm{d}\gamma_s, \tau_s(1-s)\cdot)$ and the $\mathrm{d}x_{j'}$ part is $\int_0^1 \omega(\mathrm{d}\gamma_s, \tau_s s \tau_1^{-1}\cdot)$. We deduce that $\mathrm{d}F$ checks the Nualart–Pardoux conditions.

We have therefore the following theorem.

Theorem 3.2. *The stochastic Witten complex $\mathrm{d} + \mathrm{d}F$ is continuous over the space $N.P_\infty$ of forms smooth in the Nualart–Pardoux sense.*

Let us now define the algebraic counterpart of this complex, called the Hochschild–Witten complex. Let $\Omega(L)$ be the set of forms over T_L and $\Omega(L')$ be the set of forms over $T_{L'}$. Let $\Omega_.(M)$ the set of forms over M of degree larger than 1 over M.

Let $\Omega(L) \otimes \Omega_.(M)^{\otimes n} \otimes \Omega(L')$. On of each element of this tensor product, we consider the Sobolev–Hilbert space $\|\cdot\|_{k,2}$ defined by

$$\|\omega\|_{k,2} = \|(\mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d} + 2)^k \omega\|_{L^2} \tag{3.11}$$

and we consider as tensor product the tensor product of Hilbert spaces.

If $\tilde{\omega} = \sum \tilde{\omega}_n$ where $\tilde{\omega}_n \in \Omega(L) \otimes \Omega_.(M)^{\otimes n} \otimes \Omega(L')$, we put if $z > 0$,

$$\|\tilde{\omega}\|_{z,k}^2 = \sum \frac{z^n}{n!} \|\tilde{\omega}_n\|_{k,2}^2. \tag{3.12}$$

The space of smooth Hochschild elements is given by the intersection of the Hilbert spaces given by the norms $\|\cdot\|_{z,k}^2$. We call A_∞ this space. It is a Sobolev double bar construction [MC].

We define $b_p = b_{0,p} + b_{1,p}$, where

$$\begin{aligned} b_{o,p}\tilde{\omega}_n = {} & \mathrm{d}\omega_0 \otimes \omega_1 \otimes \cdots \otimes \omega_{n+1} \\ & - \sum_{1 \le i \le n+1} (-1)^{\epsilon_i - 1} \omega_0 \otimes \omega_1 \otimes \cdots \otimes d\omega_i \otimes \cdots \otimes \omega_{n+1}, \end{aligned} \tag{3.13}$$

where $\epsilon_i = \deg \omega_0 + \sum_{1 \le j \le i} \deg \omega_j - 1$ and $\tilde{\omega}_n = \omega_0 \otimes \cdots \otimes \omega_{n+1}$. $b_{1,p}$ is defined by

$$\begin{aligned} b_{1,p} = {} & \omega_0 \wedge \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_{n+1} \\ & - \sum_{1 \le i \le n} (-1)^{\epsilon_i} \omega_0 \otimes \cdots \otimes \omega_i \wedge \omega_{i+1} \otimes \cdots \otimes \omega_{n+1}. \end{aligned} \tag{3.14}$$

Moreover, $\mathrm{d}F$ is the Chen form associated with ω. We consider the shuffle product $\omega.\tilde{\omega}$

$$\omega.\tilde{\omega} = \omega_0 \bigotimes \left(\sum_{i \le i \le n} \operatorname{sign} \omega_1 \otimes \cdots \omega_i \otimes \omega \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n \right) \bigotimes \omega_{n+1}. \quad (3.15)$$

The sign arises from the anticommutation relation over forms; let us recall that the degree of ω_0 and the degree of ω_{n+1} are kept in this formalism and that the degree of the other forms is substracted from one unit. Let us recall, moreover, that the shuffle product is continuous [L6, Theorem IV.1], and that $\tilde{\omega} \to \omega.\tilde{\omega}$ is continuous over the Hochschild space A_∞.

Moreover, since ω is a symplectic form, $\mathrm{d}\omega = 0$. We deduce that

$$b_p \omega. + \omega.b_p = 0. \quad (3.16)$$

We give the following definition:

Definition 3.3. $b_p + \omega.$ is called the Hochschild–Witten complex over A_∞.

It is a complex because $b_p^2 = 0$, $\omega.\omega. = 0$ and (3.16).

The techniques of [L5] show the following result.

Theorem 3.4. *The Hochschild–Witten complex is continuous over A_∞.*

Let Σ the map Chen iterated integral:

$$\begin{aligned} &\Sigma(\omega_0 \otimes \omega_1 \otimes \cdots \otimes \omega_n \otimes \omega_{n+1}) \\ &\quad = \omega_0(\gamma_0) \wedge \int\limits_{0 < s_1 < \cdots < s_n < 1} \omega_1(\mathrm{d}\gamma_{s_1}, \cdot) \wedge \cdots \wedge \omega_n(\mathrm{d}\gamma_{s_n}, \cdot) \wedge \omega_{n+1}(\gamma_1). \end{aligned} \quad (3.17)$$

We have

$$\Sigma b_p = \mathrm{d}\Sigma \quad (3.18)$$

and

$$\Sigma \omega_. = \mathrm{d}F \wedge \Sigma. \quad (3.19)$$

By a proof similar to [L5, Theorem II.1], we have the following result.

Theorem 3.5. *Σ is continuous from A_∞ into $N.P_\infty$.*

Remark. $\exp[F]$ does not belong to all the Sobolev spaces. So it is not clear that the cohomology of $N.P_\infty$ for $\mathrm{d} + \mathrm{d}F\wedge$ is equal to the cohomology of $N.P_\infty$ for d, although $\mathrm{d} + \mathrm{d}F\wedge$ is formally d by using the scalar gauge transform $\exp[F]$.

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